Cosmological perturbations in the inflationary Universe

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Previously defined covariant and gauge-invariant perturbation variables, representing, e.g., the fractional spatial energy density gradient on hypersurfaces of constant expansion, are used to simplify the linear perturbation analysis of a classical scalar field. With the help of conserved quantities on large scales we establish an exact first-order relation between comoving fluid energy density perturbations at 'reentry' into the horizon and corresponding scalar field energy density perturbations at the first Hubble scale crossing during an early de Sitter phase of a standard inflationary scenario.

I. INTRODUCTION

Inflationary cosmology tries to trace back the origin of structures in the Universe to quantum fluctuations of a scalar field during an early de Sitter phase (see, e.g., [1–4]). Like all length scales the corresponding perturbation wavelengths are stretched out tremendously in this period and may become larger than the Hubble length which is constant at that time. After the inflation has finished, i.e., when the universe has entered a standard Friedmann-Lemaître-Robertson-Walker (FLRW) stage, the wavelength of the large-scale perturbation increases less than the Hubble length, resulting in an inward crossing of the latter which at this time represents the particle horizon. The first problem to tackle in this scenario is the quantum theoretical characterization of vacuum fluctuations of the scalar field in the de Sitter phase (see, e.g., [1–4]). The second one is the transition from the quantum to the classical realm, i.e., the transformation of quantum fluctuations into classical energy density perturbations [5,6]. In this paper we are interested in the classical, general relativistic aspects of cosmological perturbation theory. We will therefore neither be concerned with the quantum generation nor with the transition to the classical stage. We shall assume that the perturbations behave classically once they have left the "Hubble horizon".

According to the standard inflationary picture the presently observed large-scale inhomogeneities have been beyond the "Hubble horizon" since the time at which the corresponding perturbation wavelengths crossed the Hubble scale under the "slow-roll" conditions of a scalar field driven inflationary phase. The large-scale fluctuations now carry an imprint of the fluctuations then. While this general picture meanwhile has become common wisdom, the establishing of a detailed link between these two far remote cosmological periods is anything but obvious. Therefore, the question of matching perturbations between cosmological epochs with different equations of state is of interest (see, e.g., the recent debate in [7–11]). It is the main purpose of this paper to provide a (hopefully) transparent picture of this connection within a covariant and gauge-invariant perturbation approach. This will be achieved by using previously defined covariant and gauge-invariant perturbation variables representing, e.g., the fractional, spatial energy density gradient on hypersurfaces of constant expansion, which allows us to characterize conserved quantities on large scales in a spatially flat universe in a simple way. We will find exact first-order relations between large-scale conservation quantities and comoving energy density gradients at far remote cosmological epochs, leading to definite expressions of the comoving fluid energy density perturbations at "reentry" into the horizon in terms of corresponding energy density perturbations of the scalar field in the inflationary phase.

Our perturbation analysis does not explicitly introduce perturbations of the metric tensor but is on the line of the "fluid-flow approach" used by Hawking [12], Olson [13], Lyth [14], Lyth and Mukherjee [15], Lyth and Stewart [16]. Our basic perturbation variables are covariant and gauge-invariant quantities of the type introduced by Ellis and Bruni [17] and Jackson [18], suitably generalized in [19].

Together with the stretching of all cosmic distances during the de Sitter stage, a kind of smoothing effect, sometimes called "cosmological no-hair theorem" has been conjectured [20–23], stating that "any locally measurable perturbation about the de Sitter metric is damped exponentially..." ([23]). At the first glance this seems to be contradictory to the above-mentioned, well-known existence of conserved perturbation quantities on large scales [24]. We shall clarify, which of the covariant perturbation quantities are exponentially damped, which are conserved and how these different

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quantities are related. Especially, we find that the (during the de Sitter phase) exponentially decaying perturbations cannot be neglected but start to grow again as soon as the universe enters the standard FLRW phase. The picture of decaying and resurrecting large-scale perturbations is shown to be completely equivalent to the alternative description in terms of conserved quantities.

The paper is organized as follows. In section II we introduce the basic scalar field dynamics. Section III recalls, in a fluid picture, the cosmological perturbation dynamics in terms of the Ellis-Bruni-Jackson variables [17,18]. In section IV we simplify the scalar field perturbation analysis by rewriting the relevant equations in terms of covariant and gauge-invariant perturbation quantities defined with respect to hypersurfaces of constant expansion (constant Hubble parameter). In section V we find the gauge-invariant and covariant connection between large-scale scalar field perturbations during an early de Sitter phase and corresponding fluid energy density perturbations at "reentering" the horizon during the subsequent FLRW period. The final section (Section VI) summarizes the conclusions of the paper.

II. SCALAR FIELD DYNAMICS

The early Universe is assumed to be described by the energy momentum tensor T_{mn} of a minimally coupled scalar field ϕ

$$T_{mn} = g^a_m g^b_n \phi_{,a}\phi_{,b} - g_{mn}\left(\frac{1}{2}g^{ij}\phi_{,i}\phi_{,j} + V(\phi)\right) . \tag{1}$$

Assuming furthermore $\phi_{,a}$ to be timelike, one may define a unit 4-vector

$$u_i \equiv -\frac{\phi_{,i}}{\sqrt{-g^{ab}\phi_{,a}\phi_{,b}}} \tag{2}$$

with $u^i u_i = -1$. It is well-known [25] that the expression (1) may be rewritten in the perfect fluid form

$$T_{mn} = \rho u_m u_n + p h_{mn} \tag{3}$$

with $h^{mn} = g^{mn} + u^m u^n$, the energy density

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) , \tag{4}$$

where $\dot{\phi} \equiv \phi_{,a}u^a$, and the pressure

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) . \tag{5}$$

Obviously one has

$$\psi \equiv \dot{\phi} = \sqrt{-g^{ab}\phi_{,a}\phi_{,b}} . \tag{6}$$

The scalar field dynamics may be written as

$$\dot{\rho} = -\Theta(\rho + p) \tag{7}$$

with

$$\Theta \equiv u^i{}_{;i} , \tag{8}$$

and

$$(\rho + p)\dot{u}^m = -p_{,k}h^{mk} , \tag{9}$$

where $\dot{u}^m \equiv u^m_{;n}u^n$. Since the vorticity $\omega_{ab} = h^c_a h^d_b u_{[c;d]}$ vanishes for a scalar field, the Raychaudhuri equation for the case of a vanishing cosmological constant becomes

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2\sigma^2 - \dot{u}^a_{;a} + \frac{\kappa}{2}(\rho + 3p) = 0 \ , \tag{10}$$

where

$$\sigma^2 \equiv \frac{1}{2}\sigma_{ab}\sigma^{ab} \ , \qquad \sigma_{ab} = h^c_a h^d_b u_{(c;d)} - \frac{1}{3}\Theta h_{ab} \ . \tag{11}$$

The 3-curvature scalar of the hypersurfaces orthogonal to u^a is

$$\mathcal{R} = 2\left(-\frac{1}{3}\Theta^2 + \sigma^2 + \kappa\rho\right) \ . \tag{12}$$

We recall that all the relations (7) - (12) are valid both for a scalar field and for a conventional, irrotational perfect fluid.

III. PERTURBATION THEORY ON COMOVING HYPERSURFACES

Introducing a length scale S by

$$\frac{1}{3}\Theta \equiv \frac{\dot{S}}{S} \ , \tag{13}$$

we define the comoving spatial gradient of the expansion according to

$$t_a \equiv S h^c_a \Theta_{,c} \ . \tag{14}$$

Following [18], the density inhomogeneities will be characterized by the quantity

$$D_a \equiv \frac{S h^c_a \rho_{,c}}{\rho + p} \ , \tag{15}$$

being the comoving fractional spatial gradient of the energy density. Likewise, the pressure perturbations are described by

$$P_a \equiv \frac{S h^c_a p_{,c}}{\rho + p} \ , \tag{16}$$

which allows us to write Eq.(9) as

$$S\dot{u}_m = -P_m \ . \tag{17}$$

For later use we also define the 3-curvature perturbation variable

$$r_a \equiv S h^c_a \mathcal{R}_{,c} \ . \tag{18}$$

Differentiating Eq.(7) and projecting orthogonal to u_a yields

$$h^a_n \dot{D}_a + \frac{\dot{p}}{\rho + p} D_n + \sigma^c_n D_c + t_n = 0 \ . \tag{19}$$

Analogously, one obtains from Eq.(10),

$$\begin{aligned} h^a_n \dot{t}_a = & -\dot{\Theta} P_n - \sigma^c_n t_c - \frac{2}{3}\Theta t_n - 2 S h^c_n \left(\sigma^2\right)_{,c} \\ & + S h^c_n \left(\dot{u}^a_{;a}\right)_{,c} - \frac{\kappa}{2}(\rho + p)\left[D_n + 3P_n\right] \ . \end{aligned} \tag{20}$$

For the case of vanishing vorticity the set of equations (19) and (20) is completely general, i.e., it holds for scalar fields and conventional perfect fluids (cf. Eqs.(29) and (30) in [18]). Let us now assume the spatial gradients, i.e., the inhomogeneities, as well as σ to be small. This is equivalent to assuming an almost homogeneous and isotropic

universe. We regard the quantities D_a, P_a, t_a and σ as small first-order quantities on a homogeneous and isotropic background characterized by the zeroth-order relations (superscript "0")

$$\kappa\rho^{^{(0)}} = \frac{1}{3}\left(\Theta^{^{(0)}}\right)^2 + \frac{1}{2}\mathcal{R}^{^{(0)}} \ , \tag{21}$$

and

$$\dot{\Theta}^{^{(0)}} + \frac{3}{2}\kappa\left(\rho^{^{(0)}} + p^{^{(0)}}\right) = \frac{1}{2}\mathcal{R}^{^{(0)}} \ , \tag{22}$$

with

$$\mathcal{R}^{^{(0)}} = \frac{6k}{a^2} \ , \tag{23}$$

where a is the scale factor of the Robertson-Walker metric. In linear order in the inhomogeneities the acceleration term in (20) reduces to

$$Sh^c_n\left(\dot{u}^a_{;a}\right)_{,c} = -\frac{\nabla^2}{a^2}P_n \ . \tag{24}$$

Spatial differentiation of Eq.(12) yields, in linear order, the following relation between the first-order quantities r_a, t_a and D_a

$$r_a = -\frac{4}{3}t_a + 2\kappa\left(\rho + p\right)D_a \ . \tag{25}$$

(Since the coefficients of $r_{_a}$, $t_{_a}$, $D_{_a}$, and $P_{_a}$ are always of zeroth order in a linear theory, we omit the superscript "0" from now on).

Taking into account the zeroth-order relation $h^n_0 = 0$, the system of equations (19) and (20) becomes, in linear order in the spatial gradients,

$$\dot{D}_\mu + \frac{\dot{p}}{\rho + p}D_\mu + t_\mu = 0 \ , \tag{26}$$

and

$$\dot{t}_\mu = -\frac{2}{3}\Theta t_\mu - \frac{\kappa}{2}\left(\rho + p\right)D_\mu - \left(\frac{1}{2}\mathcal{R} + \frac{\nabla^2}{a^2}\right)P_\mu \ . \tag{27}$$

The system of equations (26) and (27) is valid for a perfect fluid with arbitrary equation of state including a scalar field. In order to reformulate the dynamics for the latter case in terms of suitable scalar field quantities we realize that [26]

$$h^c_a\phi_{,c} = 0 \ . \tag{28}$$

A suitable scalar field quantity to characterize spatial inhomogeneities is [26]

$$\chi_a \equiv \frac{Sh^c_a\psi_{,c}}{\psi} \ . \tag{29}$$

Because of the relations (4), (5), (15), (16) and (28) we have (the superscript "s" denotes a scalar field quantity)

$$D^{(s)}_a = P^{(s)}_a = \chi_a \ , \tag{30}$$

and for a scalar field the system of equations (26) and (27) may be written as

$$\dot{\chi}_\mu + \left(\Theta + 2\frac{\dot{\psi}}{\psi}\right)\chi_\mu + t_\mu = 0 \ , \tag{31}$$

and

$$\dot{t}_\mu = -\frac{2}{3}\Theta t_\mu - \left[\frac{\kappa}{2}\psi^2 + \frac{1}{2}\mathcal{R} + \frac{\nabla^2}{a^2}\right]\chi_\mu \ , \tag{32}$$

respectively. It is possible to define a formal sound velocity according to

$$c_s^2 \equiv \frac{\dot{p}}{\dot{\rho}} = -\left(1 + \frac{2\dot{\psi}}{\Theta\psi}\right) = 1 + \frac{2V'}{\Theta\psi} \ . \tag{33}$$

Equations (31) and (32) may be combined to yield a second-order differential equation for χ_μ:

$$\begin{aligned} &\ddot{\chi}_\mu + \left(\frac{2}{3} - c_s^2\right)\Theta\dot{\chi}_\mu \\ &- \left[\Theta\left(c_s^2\right)^{\cdot} + \frac{\kappa}{2}\left(\rho - 3p\right)c_s^2 + \frac{\kappa}{2}\left(\rho + p\right) + \frac{\mathcal{R}}{2}\left(1 - c_s^2\right) + \frac{\nabla^2}{a^2}\right]\chi_\mu = 0 \ . \end{aligned} \tag{34}$$

For comparison we write down the corresponding equation for the fractional energy density gradient of a perfect fluid (superscript "f"), $D_\mu^{(f)}$:

$$\begin{aligned} &\ddot{D}_\mu^{(f)} + \left(\frac{2}{3} - v_s^2\right)\Theta\dot{D}_\mu^{(f)} \\ &- \left[\Theta\left(v_s^2\right)^{\cdot} + \frac{\kappa}{2}\left(\rho - 3p\right)v_s^2 + \frac{\kappa}{2}\left(\rho + p\right) + v_s^2\frac{\nabla^2}{a^2}\right]D_\mu^{(f)} = 0 \ . \end{aligned} \tag{35}$$

The quantity v_s is the adiabatic sound velocity of the fluid, determined by $v_s^2 = (\partial p/\partial\rho)_{adiabatic}$. The equations (34) and (35) are very similar. The differences are due to the different relations between the pressure perturbations and the energy density perturbations in both cases. While according to (30) the quantities D_a and P_a coincide in the case of a scalar field the corresponding relation for a fluid is [19]

$$P_a^{(f)} = v_s^2 D_a^{(f)} \ . \tag{36}$$

In terms of the potential and its derivatives, equation (34) becomes

$$\begin{aligned} &\ddot{\chi}_\mu - \frac{1}{3}\left(1 + \frac{6V'}{\Theta\psi}\right)\Theta\dot{\chi}_\mu \\ &- \left[2V'' + 2\kappa V + \frac{2V'}{\Theta\psi}\left(5 + \frac{3V'}{\Theta\psi}\right)\frac{\Theta^2}{3} + \frac{\nabla^2}{a^2}\right]\chi_\mu = 0 \ . \end{aligned} \tag{37}$$

IV. LINEAR PERTURBATION THEORY ON HYPERSURFACES OF CONSTANT EXPANSION

All the first-order quantities D_a, P_a, χ_a, t_a, r_a, have physical interpretations on comoving hypersurfaces. It was shown in [19] that the perfect fluid dynamics simplifies if written in terms of covariant and gauge-invariant variables with physical interpretations on hypersurfaces of constant curvature, constant expansion, or constant energy density. These variables correspond to certain combinations of the Ellis-Bruni-Jackson variables. E.g., the quantity

$$D_a^{(ce)} \equiv D_a - \frac{\dot{\rho}}{\rho + p}\frac{t_a}{\dot{\Theta}} \tag{38}$$

represents in first order the fractional, spatial gradient of the energy density on hypersurfaces of constant expansion (superscript "ce"). This interpretation becomes obvious if one writes down the first-order expressions for $h_a^c\rho_{,c}$ and $h_a^c\Theta_{,c}$ occuring in D_a and t_a (see the definitions (15) and (14)), respectively:

$$\left(h_0^c\rho_{,c}\right)\hat{} = 0 \ , \qquad \left(h_\alpha^c\rho_{,c}\right)\hat{} = \hat{\rho}_{,\alpha} + \hat{u}_\alpha\dot{\rho}^{(0)} \ , \tag{39}$$

and

$$(h^c_0 \Theta_{,c})\hat{} = 0 \ , \qquad (h^c_\alpha \Theta_{,c})\hat{} = \hat{\Theta}_{,\alpha} + \hat{u}_\alpha \dot{\Theta}^{^{(0)}} \ , \tag{40}$$

where the superscripts "0" again denote the homogeneous and isotropic zeroth order while the quantities with carets are of first order in the deviations from homogeneity and isotropy. Using the relations (14), (15), (39), and (40) in Eq.(38), the quantity $D^{(ce)}_a$ becomes, in first order,

$$\hat{D}^{(ce)}_a = \frac{S\hat{\rho}_{,\mu}}{\rho^{^{(0)}} + p^{^{(0)}}} - \frac{\dot{\rho}^{^{(0)}}}{\dot{\Theta}^{(0)}} \frac{S\hat{\Theta}_{,\mu}}{\rho^{^{(0)}} + p^{^{(0)}}} \ . \tag{41}$$

While $D^{(ce)}_a$, defind in Eq.(38), is an exact, covariant quantity without any reference to perturbation theory, the perturbation quantity $\hat{D}^{(ce)}_a$ is gauge-invariant by construction. Since $\hat{D}^{(ce)}_a$ is gauge-invariant, the combination on the right-hand side (RHS) of Eq.(41) is gauge-invariant as well, while each term on its own is not. Therefore, in a gauge with $\hat{\Theta}_{,\mu} = 0$ (notice that Θ itself is only determined up to a constant), i.e., vanishing perturbations of the expansion, the covariant and gauge-invariant quantity $D^{(ce)}_\mu$ coincides in first order with the fractional density gradient $S\hat{\rho}_{,\mu}/(\rho^{^{(0)}} + p^{^{(0)}})$, where ρ according to the decomposition (3) is the energy density measured by a comoving observer. A different observer, moving, e.g., with a 4- velocity n^a, normal to hypersurfaces $\hat{\Theta}_{,\mu} = 0$, would interpret a different quantity, namely $\mu = T_{ab} n^a n^b$ as energy density. (Different from the observer moving with u^a he would also measure an heat flux.) The densities ρ and μ are related by (see [27,28])

$$\mu = \rho \cosh^2 \beta + p \sinh^2 \beta \ ,$$

where $\beta(t)$ is the "hyperbolic angle of tilt" given by $\cosh \beta = -u^a n_a$. It follows that the perturbation quantity $\hat{\rho}$ for a comoving (with u^a) observer will generally not coincide with the perturbation $\hat{\mu}$, i.e., $\hat{\rho}$ will not coincide with the energy density perturbation on hypersurfaces of constant expansion. In the present case, however, u^a and n^a coincide in zeroth order and β may be considered as small ($\beta \ll 1$). Under this condition the difference between μ and ρ is of second order in β. Consequently, in linear perturbation theory we may identify the quantities $\hat{\mu}$ and $\hat{\rho}$. It follows that the quantity (41) may be interpreted as the fractional energy density gradient on constant expansion hypersurfaces in a similar sense in which D_a is interpreted as fractional energy density gradient on comoving hypersurfaces. Similar statements hold for the other perturbation variables.

A curvature variable on hypersurfaces of constant expansion, $r^{(ce)}_a$, is given by

$$r^{(ce)}_a \equiv r_a - \dot{\mathcal{R}} \frac{t_a}{\dot{\Theta}} \ . \tag{42}$$

For vanishing background curvature $\mathcal{R} = 0$ the latter quantity coincides with r_a, i.e., $r^{(ce)}_a \overset{(k=0)}{=} r_a$. In terms of the variables $D^{(ce)}_a$ and $r^{(ce)}_a$ the relation (25) reduces to [19] (cf. [33])

$$r^{(ce)}_a = 2\kappa \left(\rho + p\right) D^{(ce)}_a \ . \tag{43}$$

An expansion perturbation variable does no longer occur. Effectively, the number of variables has been reduced [19].

Analogously, for universes with nonvanishing background curvature one may define the fractional energy density on constant curvature hypersurfaces (superscript "cc")

$$D^{(cc)}_a \equiv D_a - \frac{\dot{\rho}}{\rho + p} \frac{r_a}{\dot{\mathcal{R}}} \ . \tag{44}$$

The corresponding perturbation variable of the expansion is

$$t^{(cc)}_a \equiv t_a - \dot{\Theta} \frac{r_a}{\dot{\mathcal{R}}} \ . \tag{45}$$

In terms of the variables (44) and (45), defined with respect to constant curvature hypersurfaces, the relation (25) between the three variables r_a, t_a, D_a becomes [19] (cf. [33])

$$\frac{4}{3} \Theta t^{(cc)}_a = 2\kappa \left(\rho + p\right) D^{(cc)}_a \ . \tag{46}$$

No curvature perturbation variable does occur in Eq.(46). The use of either $D^{(ce)}_a$ or $D^{(cc)}_a$ as basic variable simplifies the perturbation dynamics significantly [19]. Since for universes with zero background curvature the variable $D^{(cc)}_a$ is

not defined we will use $D_a^{(ce)}$ as independent variable in terms of which the system of equations (26) and (27) may be written in the form

$$\left[a^2\dot{\Theta}D_\mu^{(ce)}\right]^{\cdot} = 3k\Theta\left[\frac{\dot{p}}{\dot{\rho}}D_\mu - P_\mu\right] - a^2\Theta\frac{\nabla^2}{a^2}P_\mu \ . \tag{47}$$

This relation is equivalent to Eqs.(26) and (27). It contains the entire linear perturbation dynamics and holds both for a scalar field, characterized by the relations (30), and for a fluid. For a fluid equation of state $p = p(\rho)$ one has

$$\dot{p} = v_s^2\dot{\rho} \ , \tag{48}$$

and it follows with Eq.(36) that the bracket in the first term on the RHS of (47) vanishes.

For $D_\mu = D_{(n)}\nabla_\mu Q_{(n)}$, $P_\mu = P_{(n)}\nabla_\mu Q_{(n)}$ and $D_\mu^{(ce)} = D_{(n)}^{(ce)}\nabla_\mu Q_{(n)}$ (and corresponding relations for the other perturbation quantities), where the $Q_{(n)}$ satisfy the Helmholtz equation $\nabla^2 Q_{(n)} = -n^2 Q_{(n)}$ one has ([29–32]) $n^2 = \nu^2$ for $k = 0$ and $n^2 = \nu^2 + 1$ for $k = -1$, where ν is continous and related to the physical wavelength by $\lambda = 2\pi a/\nu$. For $k = +1$ the eigenvalue spectrum is discrete, namely $n^2 = m(m+2)$ with $m = 1, 2, 3.....$

It follows that for $k = 0$ the spatial gradient terms on the RHS of Eq.(47) may be neglected on large perturbation scales ($\nu \ll 1$) and the quantity $a^2\dot{\Theta}D_{(\nu)}^{(ce)}$ is a conserved quantity both for a perfect fluid and a scalar field.

In the scalar field case the curvature term on the RHS of Eq.(47) becomes

$$3k\Theta\left[\frac{\dot{p}}{\dot{\rho}}D_a^{(s)} - P_a^{(s)}\right] = -6k\left(\Theta + \frac{\dot{\psi}}{\psi}\right)\chi = -6k\frac{V'}{\psi}\chi \ . \tag{49}$$

Using the relations (38), (30), (31) and (32), we find

$$a^2\dot{\Theta}D_\mu^{(ce)} = a^2\left[\left(\dot{\Theta} - \Theta\left(\Theta + 2\frac{\dot{\psi}}{\psi}\right)\right)\chi_\mu - \Theta\dot{\chi}_\mu\right] \ . \tag{50}$$

Inserting Eqs.(49) and (50) into Eq.(47) one gets

$$\left\{a^2\left[\left(\dot{\Theta} + \Theta^2 c_s^2\right)\chi_\mu - \Theta\dot{\chi}_\mu\right]\right\}^{\cdot} = \left[6k\frac{V'}{\psi} - \Theta\nabla^2\right]\chi_\mu \ , \tag{51}$$

which is a different, more compact way of writing Eqs.(34) or (37).

It is obvious from Eqs.(43) and (22) that for $k = 0$ the relation

$$r_a^{(ce)} \stackrel{(k=0)}{=} r_a = \frac{3}{4}\left[\Theta\dot{\chi}_\mu - \left(\dot{\Theta} - \Theta\left(\Theta + 2\frac{\dot{\psi}}{\psi}\right)\right)\chi_\mu\right] \tag{52}$$

is valid. The large-scale conservation quantity (for a flat background) $a^2 r_a$ coincides with the quantity C_a used by Dunsby and Bruni [32].

V. SCALAR FIELD PERTURBATIONS IN THE DE SITTER PHASE AND THEIR RELATION TO FLUID PERTURBATIONS IN THE FLRW EPOCH

Let us now apply the equations (31) and (32), or, equivalently Eq.(51) for the general scalar field dynamics to the "slow-roll" period of an inflationary universe with $k = 0$. The "slow-roll" conditions are

$$\dot{\psi} \ll \Theta\psi \ , \qquad \Theta \approx const \ . \tag{53}$$

The relation (50) specifies to

$$a^2\dot{\Theta}D_\mu^{(ce)} = -a^2\Theta\left[\dot{\chi}_\mu + \Theta\chi_\mu\right] = -\frac{\Theta}{a}\left[a^3\chi_\mu\right]^{\cdot} \ . \tag{54}$$

¿From Eq.(47) and the discussion below Eq.(48) we know that this quantity is conserved on large scales for $k = 0$. Denoting the conserved quantity by $-E_{(\nu)}$, i.e.,

$$- E_{(\nu)} \equiv a^2 \dot{\Theta} D_{(\nu)}^{(ce)} \ , \qquad (\nu \ll 1) \tag{55}$$

equation (54) becomes

$$\left[\chi_{(\nu)} a^3\right]^{\cdot} = \frac{a}{\Theta} E_{(\nu)} \qquad (\nu \ll 1) \tag{56}$$

on large scales. Taking into account $a \sim \exp(Ht)$ with $3H \equiv \Theta$, the solution of Eq.(56) is

$$\chi_{(\nu)} = \frac{3}{\Theta^2} \frac{E_{(\nu)}}{a^2} + \frac{C_{(\nu)}^{(s)}}{a^3} \ , \qquad (\nu \ll 1) \tag{57}$$

where $C_{(\nu)}^{(s)}$ is an integration constant for the scalar field case (superscript "s"). The large-scale solution (57) is also found from the second-order equation (37) for χ_μ, which under the "slow-roll" conditions

$$\frac{V'}{\Theta \psi} \approx -1 \ , \qquad V'' \ll \frac{1}{3} \Theta^2 \ , \tag{58}$$

reduces to

$$\ddot{\chi}_{(\nu)} + \frac{5}{3} \Theta \dot{\chi}_{(\nu)} - \left[\mathcal{R} - \frac{2}{3} \Theta^2 + \frac{\nabla^2}{a^2}\right] \chi_{(\nu)} = 0 \ . \tag{59}$$

The solution (57) describes the exponential damping of the comoving, fractional energy density gradient on large scales during the de Sitter phase. It demonstrates the stability of the latter with respect to linear perturbations and seems to represent a gauge-invariant formulation of the cosmic "no-hair" theorem [20–23]. While the perturbation wavelengths are stretched out tremendously, the amplitude of the comoving, fractional energy density gradient becomes exponentially small. As will be shown below, it is *not* justified, however, to neglect these perturbations since they will resurrect during the subsequent FLRW phase. While the "hair" gets extremely shortend during the de Sitter period, it is not completely extinguished but slowly grows again afterwards.

It is of essential interest for any inflationary scenario to establish a link between the large-scale perturbations at Hubble scale crossing within an early de Sitter stage on the one hand side, and at "reentering" the horizon during the subsequent FLRW phase on the other side. It is well known that the existence of conservation quantities is helpful to find this connection [1,2,14,24,34,35]. As we shall demonstrate here the description in terms of gauge-invariant and covariant perturbation variables provides us with a definite expression for the large-scale comoving, fractional energy density gradient at "reentering" the horizon in terms of the corresponding quantity at Hubble scale crossing during the "slow-roll" period of an inflationary universe.

Let us assume the evolution of the cosmic medium to encompass an early phase during which it is adequately described by a scalar field that for some time admits the fulfillment of the "slow-roll" conditions (53) or (58). According to the standard picture (see, e.g., [4]) the scalar field finally decays and the universe enters the familiar FLRW stage which initially is radiation dominated. Since both the early scalar field dominated period and the final FLRW phase are accessible to (perfect) fluid descriptions with different equations of state, a unified, one-component picture of the cosmological evolution including an inflationary phase is possible. The different equations of state give rise to different values of the sound velocity. The time dependence of the latter is, however, fully taken into account in our basic equation (47). The set of equations (26) and (27) or, equivalently, Eq.(47), is valid both for scalar fields and for fluids.

For $k = 0$ the covariant quantity $a^2 \dot{\Theta} D_\mu^{(ce)}$ is conserved on large scales independently of the equations of state. Especially, the large-scale conservation of E_μ holds for a minimally coupled scalar field and for a conventional perfect fluid. Provided, nonadiabatic pressure perturbations during reheating do not substantially alter the present picture, on large scales E_μ remains the *same* quantity under the change from the early scalar field dominated stage to the latter FLRW epoch. Since we have established the connection between E_μ and the other covariant and gauge-invariant perturbation quantities, especially the comoving, fractional energy density gradients both for scalar fields, $D_\mu^{(s)} \equiv \chi_\mu$, and for conventional adiabatic fluids, $D_\mu^{(f)}$, we are able to express the comoving, large-scale perturbation amplitude at the time of "reentry", say, today in terms of the comoving, large-scale perturbation amplitude at the time of Hubble scale crossing during the inflationary period.

Denoting the initial Hubble scale crossing time by t_i and the time of "reentering" the horizon during the perfect fluid FLRW phase by t_e, conservation of the quantity (55) means

$$a^2(t_i)\, \dot{\Theta}(t_i)\, D_{(\nu)}^{(ce)}(t_i) \equiv -E_{(\nu)}(t_i) = -E_{(\nu)}(t_e) \equiv a^2(t_e)\, \dot{\Theta}(t_e)\, D_{(\nu)}^{(ce)}(t_e) \ , \qquad (\nu \ll 1) \ . \tag{60}$$

In order to relate $D_{(\nu)}(t_i)$ to $D_{(\nu)}(t_e)$ we have first to find the connection between $E_{(\nu)}(t_e)$ and $D_{(\nu)}(t_e)$.

Using in Eq.(55) the relations (38), (26) and (27) for $k=0$, we obtain

$$E_{(\nu)} = -a^2\dot{\Theta}D^{(ce)}_{(\nu)} = a^2\left[\Theta\dot{D}^{(f)}_{(\nu)} + \left(\frac{\gamma}{2} - v_s^2\right)\Theta^2 D^{(f)}_{(\nu)}\right] , \qquad (\nu \ll 1) \tag{61}$$

where $\gamma = (\rho^{(f)} + p^{(f)})/\rho^{(f)}$. The modes of interest are expected to "reenter" the the horizon at a period with constant values of γ and v_s^2. Under such circumstances we may write

$$E_{(\nu)} = \Theta a^2 a^{-3\left(\gamma/2 - v_s^2\right)}\left[a^{3\left(\gamma/2 - v_s^2\right)} D^{(f)}_{(\nu)}\right]^{\cdot} , \qquad (\nu \ll 1) \ . \tag{62}$$

Let us first consider the case that at "reentering" the universe is still radiation dominated, i.e., $p=\rho/3$, $\gamma=4/3$, $v_s^2 = 1/3$. In this case we have (the superscript "r" stands for radiation)

$$\left[aD^{(r)}_{(\nu)}\right]^{\cdot} = \frac{E_{(\nu)}}{\Theta a}, \qquad (\nu \ll 1) \ . \tag{63}$$

With $\Theta = 3\dot{a}/a = 3H$ and $a = a_0\left(t/t_0\right)^{1/2}$, integration of Eq.(63) yields

$$D^{(r)}_{(\nu)}(t) = D^{(r,g)}_{(\nu)}(t) + \frac{C^{(r)}_{(\nu)}}{a} , \qquad (\nu \ll 1) \tag{64}$$

where

$$D^{(r,g)}_{(\nu)}(t) = \frac{1}{9}\frac{E_{(\nu)}}{H^2a^2} , \qquad (\nu \ll 1) \tag{65}$$

is the growing mode (superscript "g") of the solution (64). The quantity $C^{(r)}_{(\nu)}$ is an integration constant for the radiation case. Because of $H^2a^2 \propto a^{-2}$ we have reproduced the well-known result for the growing ($\sim a^2$) and decaying ($\sim a^{-1}$) perturbation modes in a radiation dominated universe. The advantage of the calculation via Eq.(63) instead of solving a second-order differential equation for $D_{(\nu)}$, as is usually done, is that one obtains an explicit relation between the conserved quantity $E_{(\nu)}$ and the growing mode. *The conserved quantity $E_{(\nu)}$ is coupled to the growing mode only.* Realizing that $(Ha)^{-1}$ is the comoving horizon scale in a radiation dominated universe, Equation (65) implies that the dynamics of the growing mode $D^{(r,g)}_{(\nu)}$ is entirely determined by the square of the comoving horizon distance. It follows, that (up to a numerical constant) the conserved quantity $E_{(\nu)}$ represents the ratio of the growing part of the comoving, fractional density perturbations to the square of the comoving particle horizon.

Along the same lines one finds a corresponding relation for the scalar field in the "slow-roll" phase. Recalling $\chi_{(\nu)} \equiv D^{(s)}_{(\nu)}$ from Eq.(30), Equation (57) may be written as

$$D^{(s)}_{(\nu)} = D^{(s,d)}_{(\nu)} + \frac{C^{(s)}_{(\nu)}}{a^3} , \qquad (\nu \ll 1) \ . \tag{66}$$

$D^{(s,d)}_{(\nu)}$ is the "dominating" mode (there is no growing mode in the present case)

$$D^{(s,d)}_{(\nu)}(t) = \frac{1}{3}\frac{E_{(\nu)}}{H^2a^2} , \qquad (\nu \ll 1) \tag{67}$$

during the "slow-roll" period. *$E_{(\nu)}$ couples only to the "dominating" mode during an early de Sitter phase.* In the de Sitter period the quantity $(Ha)^{-1}$ represents the comoving event horizon, the square of which is proportional to the dominating mode $D^{(s,d)}_{(\nu)}$. In this era the constant quantity $E_{(\nu)}$ determines the ratio of the "dominating" part of the comoving, fractional energy density perturbations to the comoving Hubble horizon for $\nu \ll 1$.

The relation $E_{(\nu)}(t_i) = E_{(\nu)}(t_e)$ (cf.Eq.(60)) provides us with

$$D^{(s,d)}_{(\nu)}(t_i)\, H^2(t_i)\, a^2(t_i) = \frac{1}{3}D^{(r,g)}_{(\nu)}(t_e)\, H^2(t_e)\, a^2(t) , \qquad (\nu \ll 1) \ . \tag{68}$$

Since the horizon crossing conditions for a perturbation of the constant, comoving wavelength λ/a are $\lambda/a \approx [H(t_i)\,a(t_i)]^{-1} \approx [H(t_e)\,a(t_e)]^{-1}$, Eq. (68) reduces to

$$D^{(r,g)}_{(\nu)}(t_e) \approx \frac{1}{3} D^{(s,d)}_{(\nu)}(t_i) \qquad (\nu \ll 1)\ . \tag{69}$$

At reentry into the horizon, the comoving, fractional energy density perturbations are about 1/3 the comoving, fractional energy density perturbations at Hubble horizon crossing during the de Sitter phase. With other words, the corresponding "transfer function" is 1/3.

If the universe is already matter dominated at reentry into the horizon, Eq.(62) becomes (the superscript "m" stands for matter)

$$\left[a^{\frac{3}{2}} D^{(m)}_{(\nu)}\right]^{\cdot} = \frac{E_{(\nu)}}{\Theta a^{1/2}}\ , \qquad (\nu \ll 1)\ . \tag{70}$$

Because of $a = a_0\,(t/t_0)^{2/3}$ we find

$$D^{(m)}_{(\nu)} = D^{(m,g)}_{(\nu)} + \frac{C^{(m)}_{(\nu)}}{a^{3/2}} \qquad (\nu \ll 1)\ , \tag{71}$$

where the growing mode $D^{(m,g)}_{(\nu)}$ is given through

$$D^{(m,g)}_{(\nu)} = \frac{2}{15}\frac{E_{(\nu)}}{H^2 a^2}\ , \qquad (\nu \ll 1)\ . \tag{72}$$

The horizon crossing conditions in this case are $\lambda/a \approx [H(t_i)\,a(t_i)]^{-1} \approx 2\,[H(t_e)\,a(t_e)]^{-1}$, which yields

$$D^{(m,g)}_{(\nu)}(t_e) \approx \frac{1}{10} D^{(s,d)}_{(\nu)}(t_i) \qquad (\nu \ll 1)\ , \tag{73}$$

instead of Eq.(69). The comoving, fractional energy density perturbations at reentry into the horizon during the matter dominated FLRW period are smaller than the corresponding perturbations at Hubble horizon crossing during the "slow roll" phase by one order of magnitude, i.e., the "transfer function" is 1/10.

It becomes clear now, in which sense the exponentially decaying perturbations $D^{(s,d)}_{(\nu)}$ (cf. Eq.(67)) resurrect in order to yield the fluid energy density perturbations at reentry into the Hubble radius at $t = t_e$.

Let us denote by t_f the time at which the de Sitter phase finishes, i.e., $t_i < t_f < t_e$. During the time interval $t_f - t_i$ the scale factor increases exponentially with $H = const$. Because of the a^{-2} dependence of the "dominating" mode (67), the fractional, comoving scalar field energy density perturbations are exponentially suppressed by many orders of magnitude, i.e., $D^{(s,d)}_{(\nu)}(t_f)$ is vanishingly small compared with $D^{(s,d)}_{(\nu)}(t_i)$. For $t > t_f$ the overall energy density perturbations are represented by the fluid quantity $D^{(r,g)}_{(\nu)}$. Assuming an instantaneous transition to the radiation dominated FLRW stage, i.e., $D^{(s,d)}_{(\nu)}(t_f) \approx D^{(r,g)}_{(\nu)}(t_f)$, we have $a \propto t^{1/2}$ for $t > t_f$ and a power law growth $\propto a^2 \propto t$ of the quantity $D^{(r,g)}_{(\nu)}$. Consequently, it needs a large time interval $t_e - t_f$ during which the power law growth of $D^{(r,g)}_{(\nu)}$ compensates the initial exponential decay of $D^{(s,d)}_{(\nu)}$ during the interval $t_f - t_i$. Finally, at $t = t_e$, i.e., at the time at which the perturbation crosses the horizon inwards, the fluid energy density perturbations $D^{(r,g)}_{(\nu)}$ or $D^{(m,g)}_{(\nu)}$ are almost of the same order again as the scalar field perturbations $D^{(s,d)}_{(\nu)}$ at the first horizon crossing time t_i.

Obviously, the description in terms of comoving energy density perturbations that initially become exponentially small but afterwards start to grow again is completely equivalent to the characterization with the help of conserved quantities. It is the advantage of our covariant approach that it is capable of providing us with a clear and transparent picture of the different aspects of the behaviour of large-scale perturbations in an inflationary Universe.

VI. SUMMARY

Within a covariant and gauge-invariant approach we have investigated first-order perturbations in the inflationary universe. Perturbation quantities at the time of horizon entry during the FLRW phase were related to corresponding

quantities at the Hubble scale crossing time in an early de Sitter phase. The perturbation dynamics was simplified by using a covariant, large-scale conservation quantity, expressed in terms of energy density perturbations on hypersurfaces of constant expansion. The relations between this conserved quantity and covariantly defined, comoving, fractional energy density perturbations were clarified for different periods of the cosmological evolution. The conservation quantity turned out to be uniquely linked to the dominating comoving perturbation modes both in the "slow-roll" period with exponential expansion and in the FLRW stage. Equivalent to the characterization in terms of conserved quantities, the large-scale dynamics may be described with the help of comoving, fractional energy density perturbations. The latter are exponentially suppressed during the de Sitter phase but resurrect as soon as the universe enters the FLRW period. The corresponding "transfer functions" are 1/3 in the radiation dominated FLRW phase and 1/10 in the dust era.

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